Global Optimization of Robust Truss Topology via Mixed 0–1 Semidefinite Programming

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Outline

- Explain our problem.
- Formulate to 0–1 MISDP (Mixed Integer Semidefinite Programming).
- Show numerical examples.
Problem

- Find the optimal truss topology of robust structure. (In view of compliance)
- "Robust structure" means...
  A structure satisfying constraint conditions even if unexpected uncertain loads are applied to each node.

(ex.) Not robust

(ex.) Robust
Ground Structure Method

- One of the structural optimization method.
- STEP 1. Prepare potential members.
- STEP 2. Change cross-sectional areas to obtain the optimum solution. Some members are removed and its topology changes.
Formulation to 0–1 Mixed Integer Semidefinite Programming
Semidefinite Programming

- Semidefinite Programming
  \[
  \min \quad C \bullet X, \quad \text{Linear objective function}
  \]
  \[
  \text{s.t.} \quad A_i \bullet X = b_i, \quad \text{Linear constraints}
  \]
  \[
  X \succeq O. \quad \text{Semidefinite constraints}
  \]

- Semidefinite Condition is ...
  \[
  X \succeq O \iff \forall \psi, \quad \psi^\top X \psi \geq 0.
  \]

- Semidefinite programming can be solved efficiently by interior point method.
Optimization Problem

- Given: Ground structure, main load
- Objective function:
  - Minimization of compliance. (Maximize stiffness)
- Constraints:
  - Lower bounds for bar cross-sectional areas.
  - Upper bounds for total volume.
- Robust model
Optimization Problem

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- Robust model
Constraints

- Lower bounds for cross-sectional area $a_i$. In order to avoid member breaking and buckling.

$$a_i = 0 \text{ or } a_{\text{min}} \leq a_i.$$
Constraints

\[ a_i = 0 \text{ or } a_{\text{min}} \leq a_i, \]

\[ \iff \]

\[ a_{\text{min}} t_i \leq a_i \leq M t_i, \quad t \in \{0, 1\}^m. \]

\[
\begin{cases}
  t_i = 0 \Rightarrow 0 \leq a_i \leq 0. \\
  t_i = 1 \Rightarrow a_{\text{min}} \leq a_i \leq M.
\end{cases}
\]

\( M \) is a large number.

\( t_i \): an indicator of member existence.
Optimization Problem

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Robust model

External load = Main load + Uncertain loads.

\[ Q = (\bar{f}, r\mathbf{v}_1, r\mathbf{v}_2, \ldots, r\mathbf{v}_{n-1}) \]

\[ \bar{F} = \{ Q\mathbf{e} \mid \|\mathbf{e}\| \leq 1 \} : n - \text{dimensional ellipsoid} \]

\[ \mathbf{v}_k \] : Orthogonal complement of \( \bar{f} \).

\( r \): level of uncertainty.

Choose \( \mathbf{f} \in \bar{F} \).
Defect

- We apply uncertain loads to **EVERY** node.
- The topology can not change drastically.

× Not allowed.
Improve robust model

- Let some elements of $f$ to be 0.
- Use diagonal matrix $\text{diag}(p)$. 

$$
\begin{pmatrix}
1 & 1 \\
& & \\
& & 0 \\
& & \\
& & \\
& & \\
& & \\
& & \cdots \\
& & \\
\end{pmatrix}
\begin{pmatrix}
f_1 \\
\vdots \\
0 \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
p_j \\
\vdots \\
0 \\
\vdots \\
\vdots \\
\vdots \\
f_n \\
\end{pmatrix}
$$

$p_j$: an indicator of node existence
Condition of $p_j$

\[
\begin{cases}
    p_j = 1 : \text{A node which at least one member is connected to.} \\
    p_j = 0 : \text{Other nodes.}
\end{cases}
\]
Condition of $p_j$

\[
\begin{align*}
  p_j &= 1 : \text{A node which at least one member is connected to.} \\
  p_j &= 0 : \text{Other nodes.} \\
\end{align*}
\]

\[
\mathcal{I}_j : \text{the set of indices of members which connect to the } j\text{th node.}
\]
Condition of $p_j$

The condition of $p_j$ is equivalently written as

$$\begin{cases} 0 \leq p_j \leq 1, \\ p_j \geq t_i. \end{cases} \quad (\forall i : i \in I_j)$$

Proof.

$\exists i : t_i = 1, i \in I_j \Rightarrow 0 \leq p_j \leq 1, p_j \geq 1 \iff p_j = 1.$

Otherwise, no member is connected to the $j$th node.

If $p_j \neq 0$, the $j$th node is loaded. $\cdots$ Infeasible.

Thus, automatically $p_j = 0$. $\square$
Robust model

External load = Main load + Uncertain loads.

\[ Q = (\bar{f}, rv_1, rv_2, \ldots, rv_{n-1}) \]

\[ \mathcal{F}(p) = \{ \text{diag}(p)Qe \mid \|e\| \leq 1 \} \]

Choose \( f \in \mathcal{F}(p) \).
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Objective function

- Compliance

\[ c(a; f) = f^\top u \]
\[ = \sup_{u \in \mathbb{R}^n} \{2f^\top u - u^\top K(a)u\} \]

- Minimization of the compliance of the worst case.

\[ \min \tau, \quad \text{s.t.} \quad \tau \geq \sup \{c(a; f) \mid f \in \mathcal{F}(p)\}. \]
Lemma

\[ \tau \geq \sup_{\mathbf{f}} \{ c(a; \mathbf{f}) \mid \mathbf{f} \in \mathcal{F}(p) \} \]

\[
\begin{pmatrix}
\tau I & (\text{diag}(p)Q)^\top \\
\text{diag}(p)Q & K(a)
\end{pmatrix} \succeq O.
\]

Immediately obtained from a Lemma of 

Formulate into 0–1 MISDP

\[
\begin{align*}
\min & \quad \tau, \\
\text{s.t.} & \quad \begin{pmatrix} \tau I & (\text{diag}(\mathbf{p})\mathbf{Q})^\top \\ \text{diag}(\mathbf{p})\mathbf{Q} & K(\mathbf{a}) \end{pmatrix} \succeq 0, \\
& \quad a_{\min}t_i \leq a_i \leq Mt_i, \\
& \quad \mathbf{a}^\top \mathbf{l} \leq \bar{V}, \\
& \quad 0 \leq p_j \leq 1, p_j \geq t_i, \quad (\forall i : i \in I_j) \\
& \quad t \in \{0, 1\}^m.
\end{align*}
\]
Features

- 0–1 variables are only $t$.
- How to solve it?
  - Make a relaxed problem by branch-and-bound method.
  - Solve the relaxed problem by interior point method.
Numerical examples

22-member truss \((r = 0.1)\)

Ground structure

Optimal solution
Numerical examples

51-member truss \((r = 0.1)\)

Ground structure

Optimal solution
Relation between $r$ and topology

- What will happen if we change the level of uncertainty?
- Test set: 22-member truss.
  Level of uncertainty $0 \leq r \leq 1$.
- Compare the compliance, the level of uncertainty and the topology.
Relation between $r$ and topology

- Compliance is in inverse proportion to structural volume.
- For the same performance, the structure of small compliance needs only small volume (i.e. little costs).
- The graph shows a trade-off relation between level of uncertainty and the optimal cost of a structure.
Conclusion

- We formulate a mathematical programming for truss topology optimization under load uncertainties.
- We investigated on a relation between the level of uncertainty and the topology.